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THE DETERMINATION OF COMPENSATION FUNCTIONS FOR LINEAR FEEDBACK SYSTEMS TO PRODUCE SPECIFIED CLOSED-LOOP POLES

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INTRODUCTION

At present, the design of compensation networks for linear feedback control systems is very often more of an art than a science. To those somewhat unfamiliar with the field, there may seem to be a certain amount of sorcery and black magic involved. For years, various workers have pointed out the desirability of an exact, rigorous method of synthesis. Such a method would start from the general specifications of what the system is to accomplish. It would proceed in a logical, step-by-step manner to a detailed description of each component in the system. Truxal points out the advantages such a method would have when he describes Guillemin's design procedure.

As outlined, Guillemin's procedure involved three steps:

1. The closed-loop transfer function is determined from the specifications.
2. The corresponding open-loop transfer function is found.
3. The appropriate compensation networks are synthesized.

Unfortunately, in practice this procedure is not always easily carried out. The first difficulty is that there is not complete freedom in setting an open-loop transfer function. Many times it is highly desirable to use certain components which happen to be on hand or are easily obtainable. Now, if it could ever be shown rigorously that one must have complete freedom in setting the open-loop transfer function or else it is impossible to solve the problem, then this difficulty would be ended. Design engineers would merely tell those who control the purse strings that a component with certain characteristics is needed, and never mind the fact that we have a warehouse full of black boxes which are almost the same. However, there are too many instances where an engineer had to make do with what he had, and succeeded brilliantly with a trial-and-error technique, for such a story to be believed. This points up the second big difficulty with the procedure outlined above.

The fact that there is not complete freedom in setting the open-loop transfer function means that there is not complete freedom in setting the closed-loop transfer function. In turn, this may mean that a proposed closed-loop response

which is determined in step one above may be incompatible with the components which are already in the system. Since it is possible to succeed in spite of this incompatibility by coming up with a system which fulfills the requirements of the general specifications on the system, this shows that the present methods of translating general system specifications into closed-loop transfer functions are not perfectly satisfactory. Often, any system out of a large class of possible systems would turn out to be capable of doing the job which is required to be done, in a perfectly acceptable manner. What I am saying is that you have to know what you want before you start. Sometimes in an effort to pin things down precisely, one makes the mistake of pinning down the wrong thing too precisely. Much later, it is discovered that a certain specification can be relaxed with no detriment of system performance.

This paper proposes a new approach to the philosophy of design of feedback systems. In the past, most techniques would attempt to start with open-loop characteristics and work towards closed-loop or visa versa. Usually some sort of step-by-step procedure was required, involving trial-and-error or one operation after another in some long sequence. Difficulties arose because the results of any one step might require backing up and modification of an earlier step. The new approach suggested here would avoid this by doing everything in one shot. Write down exactly what job you want the system to do. Include everything which is essential, but don't put in any requirements which are arbitrary or unnecessary. Write down the characteristics of the components you have available to do the job. Now translate everything you have written down into mathematical equations. You now have a set of simultaneous equations describing the requirements on the system and the components available. Provide sufficient variables in the system parameters so that the number of variables equals the number of equations. Now solve the set of equations simultaneously and the job is done.

It is the purpose of this paper to attempt to survey the present state of the art of the synthesis of linear feedback control systems and to point out those areas in which a number of loose ends have been left dangling. Present procedures can be improved by achieving the ability to formulate general

desired results into exact analytical relationships. Once such relationships are obtained, a digital computer can take it from there. Although the whole field still requires more study, one step in the procedure has been made much more exact by the development of a new tool.

This new tool is the ability to specify the exact location of any desired number of closed-loop poles. All of the closed-loop poles of the system may be specified, or only part of them, as desired. The necessary compensation is not determined by a trial-and-error or iterative method, but by an exact procedure involving the solution of simultaneous linear equations. The method does not require the cancellation or nullification of fixed poles and zeros already in the system. Instead, it makes use of them to aid in producing the desired closed-loop poles.

The mathematical basis of this procedure will be presented first and the exact technique will be explained. This will be followed by an example which gives a clear indication of the present state of development of this approach to synthesis. Finally, a number of possibilities which the technique opens up will be surveyed and suggestions for future development will be made.

MATHEMATICAL DEVELOPMENT

It is assumed that we are given a feedback system of the form shown in Figure 1. There is a single feedback path of unity gain. Although it may appear that this configuration is unduly restrictive, it will be shown later that this is not the case and other configurations will be considered.

Let $\frac{N(s)}{D(s)}$ be the actual transfer function of the closed-loop system. Do not specify this function exactly but leave it adjustable. Fixed elements (the so-called 'plant') are already present. Let $\frac{m(s)}{n(s)}$ be the transfer function of the plant. The desired closed-loop transfer function is given as $W(s)$. The problem is to determine the compensation and gain which will result in a closed-loop system having a transfer function which approximates $W(s)$ as closely as possible. Let $\frac{a(s)}{b(s)}$ be the transfer function of the compensation which is to be determined, including the gain.

First of all, we know that

$$W(s) = \frac{\frac{a(s)}{b(s)} \frac{m(s)}{n(s)}}{1 + \frac{a(s)}{b(s)} \frac{m(s)}{n(s)}} \qquad \text{Eq. 1}$$

Multiplying out and solving for $\frac{a(s)}{b(s)}$ gives

$$\frac{a(s)}{b(s)} = \frac{n(s)}{m(s)} \; \frac{W(s)}{1 - W(s)} \qquad \text{Eq. 2}$$

This says that if there are to be no compromises or restrictions on W(s), $\frac{a(s)}{b(s)}$ must cancel out the poles and zeros of the plant and insert new ones to provide the exact specified response. In the usual case, such a procedure is undesirable. In compromise, one is usually willing to restrict or modify W(s) so as to be compatible with the system and still approximate the exact desired response closely.

As was already pointed out, usually the specification of an exact function of s for W(s) proves to be too stringent and unnecessarily strict. Actually, there are a whole class of response functions which will all meet the real requirements on the system equally well. The problem is in translating the real requirements into some sort of specification on W(s) which is sufficiently strict without being unnecessarily so.

Repeating Equation 2,

$$\frac{a(s)}{b(s)} = \frac{n(s)}{m(s)} \; \frac{W(s)}{1 - W(s)} \qquad \text{Eq. 2}$$

In order to satisfy this equation, either b(s) or W(s) must contain m(s) as a factor. That is, the open-loop fixed zeros [factors of m(s)] must either be cancelled by compensation poles [factors of b(s)] or else must appear as closed-loop zeros [factors of the numerator of W(s)]. There is no other alternative. Since the use of cancellation is undesirable, and in fact unnecessary in view of what was said about approximation techniques, we will allow the open-loop zeros to appear as closed-loop zeros.

In order to continue, replace $W(s)$ in Equation 2 by $\frac{N(s)}{D(s)}$. This gives

$$\frac{a(s)}{b(s)} = \frac{n(s)}{m(s)} \frac{\frac{N(s)}{D(s)}}{1 - \frac{N(s)}{D(s)}} = \frac{n(s)}{m(s)} \frac{N(s)}{D(s) - N(s)} \qquad \text{Eq. 3}$$

The factors of $n(s)$ are the open-loop fixed poles. In order to satisfy Equation 3, either $a(s)$ contains $n(s)$ as a factor, which means that compensation zeros cancel plant poles, or else $N(s) - D(s)$ contains $n(s)$ as a factor. It is not necessary for $N(s)$ and $D(s)$ separately to contain $n(s)$ as a factor, but only that their difference contains it.

The fact that open-loop zeros appear as closed-loop zeros and the fact that $N(s) - D(s)$ must contain $n(s)$ as a factor are the constraints placed upon the closed-loop response by the fixed open-loop poles and zeros. As it turns out, there is still plenty of freedom in setting the closed-loop response $\frac{N(s)}{D(s)}$.

Since we have just decided that it will be acceptable for open-loop zeros to appear as closed-loop zeros, we may write

$$N(s) = a(s)m(s) \qquad \text{Eq. 4}$$

Substituting this into Equation 3 yields

$$\frac{a(s)}{b(s)} = \frac{n(s)}{m(s)} \frac{a(s)m(s)}{D(s) - a(s)m(s)} \qquad \text{Eq. 5}$$

Cancelling factors where possible leads to

$$\frac{1}{b(s)} = \frac{n(s)}{D(s) - a(s)m(s)} \qquad \text{Eq. 6}$$

or, finally, after rearranging,

$$D(s) = a(s)m(s) + b(s)n(s) \qquad \text{Eq. 7}$$

The synthesis problem now becomes one of finding the compensation $\frac{a(s)}{b(s)}$ such that Equations 4 and 7, plus other equations derived from transient- or frequency-response requirements, are satisfied. Such a set of equations might be satisfied exactly in some cases, or in a mean-square-error sense in other cases.

In order to see how this procedure may be carried out, let us assume that we do not care where the closed-loop zeros of the system are, but that we wish to specify exactly all or part of the closed-loop poles. Notice that doing this does not specify the resulting closed-loop response, because in the process of compensation additional open-loop zeros will be added to the system. These zeros will also become closed-loop zeros, and may have a significant effect on the resulting closed-loop response. There are no specifications upon the closed-loop response other than that the exact locations of the closed-loop poles are specified. Having seen how to obtain a solution for this case will give us insight into other cases where different specifications are placed on the closed-loop response.

First, provision must be made for the cases where only part of the closed-loop poles are specified, and part are left unspecified. Let $D'(s)$ be closed-loop poles whose locations are to be specified. Let $x(s)$ be the remaining closed-loop poles whose locations are undetermined at the outset.

Rewrite Equation 7 with this notation.

$$D'(s)x(s) = a(s)m(s) + b(s)n(s) \qquad \text{Eq. 8}$$

Let the number of fixed plant zeros = j = the degree of $m(s)$
Let the number of fixed plant poles = k = the degree of $n(s)$
Let the number of specified closed-loop poles = r = the degree of $D'(s)$
Let the number of unspecified closed-loop poles = d = the degree of $x(s)$
Let the number of compensation zeros = q = the degree of $a(s)$
Let the number of compensation poles = p = the degree of $b(s)$

The quantities j, k, and r are known at the outset. The quantities d, p, and q are not yet known, although some relations concerning them will be derived

shortly. All of the various factors may be multiplied out to obtain polynomials of the following form:

$$m(s) = s^j + m_{j-1}s^{j-1} + \ldots + m_1 s + m_o$$

$$n(s) = s^k + n_{k-1}s^{k-1} + \ldots + n_1 s + n_o$$

$$D'(s) = s^r + w_{r-1}s^{r-1} + \ldots + w_1 s + w_o$$

$$x(s) = s^d + x_{d-1}s^{d-1} + \ldots + x_1 s + x_o$$

$$b(s) = b_p s^p + b_{p-1}s^{p-1} + \ldots + b_1 s + b_o$$

$$a(s) = a_q s^q + a_{q-1}s^{q-1} + \ldots + a_1 s + a_o$$

The coefficients of $m(s)$, $n(s)$, and $D'(s)$ are all numerical, because the factors of these functions are all known at the outset. The coefficients of $x(s)$, $b(s)$. and $a(s)$ are all unknown, and are left assumed in literal form. Note that $a(s)$ and $b(s)$ are the only polynomials which have leading terms whose coefficients may be different from unity. These coefficients provide for the Bode gain of the compensation.

These polynomials may now be substituted into Equation 8, and the indicated multiplications actually performed. The coefficients of the various powers of s on either side of Equation 8 will now be combinations of literal and numerical terms. The coefficients of like powers of s on each side of the equation are then separately equated. This will lead to $(r + d)$ linear simultaneous equations, which may be solved for the coefficients of $a(s)$, $b(s)$, and $x(s)$. The fact that these simultaneous equations will always be linear must be emphasized. Having obtained numerical values for these coefficients, the polynomials $a(s)$, $b(s)$, and $x(s)$ may now be reconstructed and factored. These factors will be the poles and zeros of the compensation, and the remaining closed-loop poles. Thus, in this case, an exact solution is obtained for the required locations of the poles and zeros of the compensation network.

Now, notice that the highest power of s on the left-hand side of Equation 8 will be the $(r + d)$ power. Thus, there will be $(r + d + 1)$ terms on the left-hand side. Since the highest power of s will always have a coefficient of unity, no equation may be obtained from this term except in one special case. All of the rest of the terms will yield useful equations. Therefore, there will be $(r + d)$ simultaneous equations in all cases, with one exception.

There are d unknown coefficients in $x(s)$, p or $(p + 1)$ unknown coefficients in $b(s)$, and q or $(q + 1)$ unknown coefficients in $a(s)$. The total number of open-loop poles in the final system will be $(p + k)$. The total number of open-loop zeros in the final system will be $(q + j)$. The total number of closed-loop poles in the final system must equal $(p + k)$ or $(q + j)$, whichever is greater.

If $(p + k)$ is greater than $(q + j)$, the complete term on the right-hand side of Equation 8 involving the highest power of s will be $b_p s^p s^k$. The highest power of s on the left-hand side of the equation will have a coefficient of unity. Thus, whenever $(p + k)$ is greater than $(q + j)$, we must have $b_p = 1$. Since this can easily be seen at the outset, there is no point in carrying along this extra equation, which is why it was stated that there are $(r + d)$ equations instead of $(r + d + 1)$. Also, notice that in this case, the leading coefficient of $a(s)$, a_q, is still an unknown.

Similarly, if $(q + j)$ is greater than $(p + k)$, it is seen that a_q will be unity and b_p will be unknown. In this case, there are still $(r + d)$ equations.

In the special case, when $(p + k) = (q + j)$, both b_p and a_q will be unknowns. There will be an extra equation, namely, $b_p + a_q = 1$. However, in this case there is also an extra unknown.

Therefore, in all cases but one, there will be $(r + d)$ simultaneous equations and $d + p + q + 1$ unknowns. In the one special case when $(p + k) = (q + j)$, there will be $(r + d + 1)$ simultaneous equations and $d + p + q + 2$ unknowns. In any event, in order to obtain an exact solution, it is clear that it is necessary to have

$$p + q = r - 1 \qquad \text{Eq. 9}$$

This says that the total number of compensation poles plus zeros must be one less than the number of closed-loop poles which are to be specified.

Let us consider some of the implications of Equation 9. For simplicity, let us assume, as is always the case in any physical system, that the number of open-loop poles in the plant exceeds the number of plant open-loop zeros; i.e., $k > j$. Now, let us also assume, for the same reason, that the number of poles in the compensation is at least equal to or else greater than the number of compensation zeros. That is, $p \geq q$. Then, the total number of closed-loop poles will be $(p + k)$. We must have $p + k = r + d$. Suppose that we want to specify the locations of all of the closed-loop poles in the resulting system. Then, $d = 0$, and $p + k = r$. Substituting this into Equation 9 shows $p + q = p + k - 1$, or

$$q = k - 1 \qquad \text{Eq. 10}$$

The number of zeros in the compensation must be one less than the number of poles in the plant. Since it was stated above that $p \geq q$, we also have

$$p \geq k - 1 \qquad \text{Eq. 11}$$

Provided that Equations 10 and 11 are satisfied, we can always specify the exact location of every resulting closed-loop pole of the system. This is sort of a handy thing to be able to do.

In the case presently under consideration, we have placed no other restrictions on the system than that the locations of the closed-loop poles be specified. This means that the resulting compensation may turn out to be of unusual form. For instance, it may be required that compensation poles be placed in the right half of the s-plane. Of course, this can be accomplished by using active compensation networks which produce unstable poles. Since we have specified the locations of <u>all</u> the closed-loop poles of the over-all system, if these were all specified in the left-half of the s-plane, the over-all

closed-loop system will be stable. It is not the purpose of this paper to go into the old argument of whether or not such a situation is desirable.

If only part of the closed-loop poles are specified, the only relations we have to guide us are $p + q = r - 1$, and $p + k = r + d$ (still assuming that $k > j$ and $p \geq q$). Remember, r is the number of specified closed-loop poles. Within these restrictions, p and q (the number of compensation poles and zeros respectively) may be chosen anything we want. This means that there exists a variety of compensation networks having different numbers of poles and zeros, at different locations, which will all produce specified closed-loop poles in exactly the same locations. It can easily be seen that the total number of such networks is r. In all of these networks, the only thing which remains constant is the sum of the number of poles plus the number of zeros, which must equal $r - 1$. Of course, the things which change in the over-all closed-loop system are the locations of the unspecified closed-loop poles.

A digital computer routine has been set up at Space Technology Laboratories which obtains the exact solution for any problem in which the only requirements on the closed-loop system are the locations of the closed-loop poles. This digital routine has proved to be very successful, and a number of problems has been solved through its use.

PRACTICAL APPLICATION

With the digital computer routine presently in existence at Space Technology Laboratories, the only specifications which may be put on the system are the desired locations of closed-loop poles, and the number of compensation poles to be used (note that specifying the number of compensation poles also specifies the number of compensation zeros). The routine then obtains the exact solution. At present, when it is desired to place other constraints on the system (e.g., stability requirements, residues, etc.), an iterative or trial-and-error technique must be employed.

The most successful technique so far along these lines has been found to be one in which more closed-loop pole locations are initially specified than are actually desired. After the solution to this is obtained, the compensation is gradually simplified until an acceptable result is obtained. In such cases, the only requirement on some of the closed-loop poles is that they be stable. However, their exact locations must initially be specified in order to use the computer routine. It usually turns out that these closed-loop pole locations are somewhat critical in that they have a drastic effect on the compensation required. Sometimes it is necessary to make several computer runs for the same problem, varying slightly a few of the closed-loop pole locations from run to run. Usually, additional information is available at the start which aids in making a wise choice for the closed-loop pole locations initially specified. In all cases, general trends are easily observed after one or two runs. Since a problem involving fifteen or twenty closed-loop poles only requires about eighty seconds of machine time, this procedure is considered to be quite practical.

For example, look at the root-locus plot in Figure 2. This is a root-locus plot of a typical autopilot for a ballistic missile. The locations of the open-loop poles and zeros and the closed-loop poles are tabulated below the figure.

The large number of poles and zeros results from all of the dynamic effects which must be considered for an accurate study. The missile in flight behaves like a bending beam, and this gives rise to bending modes at various frequencies which lie within the pass band of the autopilot unless suitable filtering is used. The dynamics of the hydraulic actuators, the rate gyro, and various other effects must be included. A detailed description of these effects is not necessary for this example.

Incidentally, it may be of interest to mention how the open-loop poles and zeros were determined. A set of differential equations was written which includes all of the dynamic effects of importance. These equations were linearized and written in Laplace-transform form. A digital computer was

used to obtain the open-loop poles and zeros from the set of simultaneous equations. Another digital computer routine is used which makes root-locus plots automatically when given the open-loop poles and zeros. A digital computer is necessary to handle the tremendous amount of computation which is involved in the analysis of such complex systems. Similarly, the methods which this paper describes are practical only when there is recourse to a digital computer.

Of the open-loop poles and zeros in Figure 2, the majority were placed not by choice but by fate. This system must now be compensated in order to provide suitable closed-loop response. The specifications on the closed-loop response are rather broad, and take somewhat the following form: First of all, the system must be stable. Next, the bandwidth must be high enough to provide sufficiently rapid response to guidance commands and to maintain tight control in the presence of external upsetting torques, such as wind gusts. Finally, the bandwidth must be low enough so that high-frequency noise is no problem.

In Figure 2, the dominant closed-loop poles are the conjugate complex pair at $s = -1.62 + j5.837$ and $s = -1.62 - j5.837$. These are termed the rigid-body poles of the system. Their location is chosen to yield acceptable transient response. The only requirement on the rest of the closed-loop poles is that they be stable. Essentially, the only compensation in the system is the zero at $s = -2.446$, resulting from the rate gyro, and the pole at $s = -23.8$, resulting from a first-order filter. These are the only critical frequencies of the system which are adjustable. Additional compensation elements must be added to obtain greater adjustment.

It can be seen that, with the given compensation, there is an unstable closed-loop pole at $s = +3.47 \pm j73.81$. This closed-loop pole comes from the first bending mode. It was determined by the method which this paper described that the rigid-body response could be improved and the first bending mode could be stabilized by using a filter containing a pair of conjugate complex poles instead of the simple single lag filter. One possible solution which appears quite acceptable is shown in Figure 3.

By moving the rate gyro zero to $s = -2.21$, removing the single lag filter, and placing filter poles at $s = -4.7 \pm j19$, the rigid body poles are moved to $-3.0 \pm j6.0$ and the first bending mode pole is moved to $s = -1.0 \pm j73.0$. The compensation was determined by the digital computer program which requires as an input the desired location of the closed-loop poles. The location of these closed-loop poles was found to be critical. If the bending mode closed-loop poles were placed at $s = -1.0 \pm j75.0$, for example, the computer revealed that it is impossible to obtain a compensation network using only two poles in the left-hand plane.

The difference in the missile response between having the first bending mode closed-loop poles at $s = -1.0 \pm j73.0$ and having them at $s = -1.0 \pm j75.0$ is negligible. Yet, what a drastic difference it makes in an attempt to determine an acceptable compensation network! This example shows the need for stating clearly at the outset what is acceptable and what is not, without over-constraining the problem.

Let us think about this example some more, to see how the transition from the unstable system in Figure 2 to the stable system in Figure 3 would be accomplished by using common present day techniques. The requirement is to place the rigid-body poles at $s = -3.0 \pm j6.0$, keeping these the dominant pair of poles in the system, and simultaneously stabilize the first bending mode without causing anything else to go unstable.

First, the design engineer would have to guess the form of the compensation. A little trial and error would reveal that the requirements can not be met with only one pole, no matter where it is placed. The next logical guess would be to use two poles, and the engineer would now start making guesses about their location. After each trial, he would make a root-locus plot to see how close he had come. With luck, the plot would reveal information which would aid in making the next guess about the location of the compensation poles. Eventually, he would converge upon a solution similar to the one shown in Figure 3. Now, admittedly, these guesses are very educated guesses, and there is a fairly systematic procedure which one can follow to insure that a minimum amount of time will be wasted. Still, the whole approach lacks a certain rigor and is not perfectly satisfying.

The procedure which was actually used in obtaining the compensation shown in Figure 3 will now be described. We may assume that Figure 2 is on hand when we start. It is seen that increased gain and phase lead is required at low frequencies in order to re-locate the rigid body poles. On the other hand, attenuation and phase lag is required at the first bending mode frequency in order to stabilize this pole. Since the only other poles which are in danger of going unstable are at high frequencies also, we may assume that they will also experience attenuation and phase lag. From examination of Figure 2, it appears that a reasonable amount of phase lag will not endanger these other poles, and attenuation can only improve the situation. Thus, it should be sufficient to specify only the desired locations of the rigid body and first bending mode closed-loop poles, and let the other closed-loop poles take care of themselves.

So far, we have made a preliminary analysis along conventional lines. It is at this point that the new approach is taken. The desired locations of the rigid body and first bending mode closed-loop poles are fed into the digital computer routine which was developed from the methods described. Since only four closed-loop pole locations are being specified, it is known immediately that a total of only three compensation elements is required (i.e., three poles, two poles, and one zero; two zeros and one pole, or three zeros). The gain and phase lead required at low frequencies can probably be obtained by relocation of the rate gyro zero, so this will be included as part of the compensation. This leave two elements. Now, we require attenuation and phase lag at higher frequencies, and we know from Figure 2 that a single pole does not provide enough. Thus, one zero and two poles appear to be the best choice.

In using the digital computer routine, one may specify all of the closed-loop poles of the system, or one may specify only part of them and let the computer find the rest. The type of compensation to be used must also be specified, so the computer is told to use two poles and one zero. The routine then computes the locations in the s-plane of these compensation elements, the Bode gain required, and the locations of all the remaining

unspecified closed-loop poles. In this case, the rigid body and first bending mode closed-loop poles were specified at the locations shown in Figure 3, and the computer gave the result to relocate the rate gyro zero where it is shown in Figure 3, and to insert two complex conjugate compensation poles where they are shown in Figure 3. This result is exact. When it was checked by using conventional root-locus techniques on the compensated system, the rigid body and first bending mode closed-loop poles turned out to lie at exactly the specified frequencies. As can be seen, all of the other closed-loop poles remained stable.

Thus, instead of requiring an iterative trial-and-error technique, only one computation leading to an exact result was needed. Although only four closed-loop poles were specified in this case, there is no limit to the number which could have been specified except the data handling capabilities of the machine. Of course, the complexity of the resulting compensation would increase accordingly.

Now, the question arises, how did we know where to specify the closed-loop poles? The question of incompatibility of specifications mentioned earlier enters here. Fortunately, it turns out that a closed-loop pole whose location is critical as far as transient response aspects of the system are concerned, will generally not be critical in determining the compensation. On the other hand, a closed-loop pole whose location is critical in its effect in determining system compensation is generally unimportant in regard to its effect on transient response. This is because closed-loop poles which are important to transient response (so-called 'dominant' poles) are usually located relatively far away from any open-loop poles or zeros of the system, and thus small variations in their position will not greatly affect the gain and phase pattern required in the s-plane. However, small variations in their positions will have a considerable effect on the time response of the system. Conversely, closed-loop poles which are unimportant to transient response are usually located nearby to open-loop poles or zeros. This means that small variations in the location of such closed-loop poles will have a radical effect on the

gain and phase pattern in the s-plane which is required to produce closed-loop poles in the specified locations. At the same time, these variations in location will have a negligible effect on the time response of the system.

Thus, we could specify that the rigid body poles occur at $s = -3.0 \pm j6.0$ with a reasonable amount of confidence that this would c e no more difficult a compensation problem than if the poles had been specified at, say, $s = -2.75 \pm j6.25$. On the other hand, as mentioned previously, it makes a tremendous difference as far as the resulting compensation is concerned exactly where we specify the first bending mode closed-loop poles. It is exceedingly desirable that the two compensation poles be located in the left half of the s-plane. It is also desirable that the closed-loop poles which result from the addition of these open-loop compensation poles have no detrimental effect on the transient response of the system. These constraints must be kept in mind as the procedure is arried out. All too often, such constraints are considered only subconsciously instead of being analytically formulated when one attempts to carry out a solution.

Being aware of this situation at the outset, the closed-loop first bending mode pole was specified to lie in a region where the closed-loop pole would occur if approximately 90° phase lag and 10 db of attenuation were added at this frequency. Admittedly, this step involves an element of judgment, or trial-and-error, or even guesswork, and implies a certain anticipation of the compensation at the beginning of the problem. In a sense, one difficulty has been traded for another. Instead of guessing at the form of the compensation and solving for the closed-loop pole locations, we now guess at the closed-loop pole locations and solve for the compensation. However, the ability to do this proves to be a tool of major importance in the design of feedback control systems, as apparently until now there has been no way to accomplish this exactly without resorting to cancellation methods when a large number of poles and zeros are involved.

EXTENSION OF PRESENT RESULTS

A number of possibilities for future development are now suggested by our present knowledge. First of all, in setting up the set of simultaneous equations, some freedom may be obtained by not requiring Equation 9 to be satisfied.

If (p + q) is less than (r - 1), it is impossible to obtain any solution, because there are more equations than there are unknowns. On the other hand, when (p + q) is greater than (r - 1), there are more unknowns than equations, and an infinite number of solutions is obtained. In this case, there is a whole locus of points which are acceptable locations for the compensation poles and zeros.

The possibilities of this last case seem most intriguing. Other external equations which express additional constraints may now be added, and a solution can still be obtained. Essentially, this says that if you use enough compensation, you can do almost anything. For instance, the residues in some of the closed-loop poles could be specified, or the position, velocity, and accelerations could be specified.

Suppose that we have specified that there be a closed-loop pole at $s = -s_o$, and we also want to require that the residue in this pole be R_o. The expression for the residue in this pole is $(s + s_o) \left. \dfrac{a(s)m(s)}{D'(s)x(s)} \right|_{s = -s_o}$

In any servo system with an open-loop pole at the origin, the velocity constant K_v is obtained from the following expression[6]:

$$\frac{1}{K_v} = -\left\{ \frac{d}{ds} \left[\log \frac{a(s)m(s)}{D'(s)x(s)} \right] \right\}_{s = o}$$

We may set the expressions for the residues in some of the poles equal to the desired values, or set the expression for the velocity constant equal to the desired value, and by including these relations with our set of simultaneous equations above, additional constraints are imposed by means

of which we can force the system to do what we want it to do. Unfortunately, as can be seen, many of these additional constraint relations turn out to be non-linear, so that solution of the resulting set of simultaneous equations becomes more difficult.

It may be noticed from Equations 10 and 11 that the number of compensation poles and zeros which must be used in order to specify the location of every closed-loop pole of the system is uncomfortably large. Many times, the exact locations of only a few of the closed-loop poles of a system are of critical importance. The requirements on the rest of the closed-loop poles will be only that they be located in some general region, or it may even be that the only requirement on some of the closed-loop poles is that they be stable. In such cases, it seems that it should be possible to reduce considerably the complexity of the compensation network required from what is indicated by Equations 10 and 11.

The problem here is one of deciding at the outset exactly what is desired, and then writing equations which will constrain the system to do what is wanted without over-constraining it. This is not always easy to do, and work is presently underway on finding suitable techniques for accomplishing this. One approach to the problem which appears promising could be based on the following principles. Suppose that the exact locations of a few of the closed-loop poles are to be specified, and the only requirement on the rest of the closed-loop poles is that they be stable. It will also be required that the poles of the compensation all be stable. The unspecified closed-loop poles are the factors of the polynomial $x(s) = s^d + s_{d-1}s^{d-1} + ... + s_1 s + x_o$. Split this polynomial into even and odd parts. For the sake of illustration, let us assume that d is an even number. Then, the even part of $x(s) = x_{even}(s) = s^d + x_{d-2}s^{d-2} + ... + x_2 s^2 + x_o$. The odd part of $x(s) = x_{odd}(s) = x_{d-1}s^{d-1} + x_{d-3}s^{d-3} + ... + x_3 s^3 + x_1 s$. It is well known that if all of the factors of x(s) are to have negative real parts, then the factors of $x_{even}(s)$ and $x_{odd}(s)$ must lie on the $j\omega$ axis and must alternate.[4] An equivalent statement is that if the ratio $\frac{x_{odd}(s)}{x_{even}(s)}$ is expanded

in a continued fraction expansion, all of the terms of the continued fraction must have the same sign.[4] These facts may provide the basis for writing constraint equations which could be included among the other simultaneous equations in setting up the problem. One interesting point is the following: If the ratio $\frac{x_{odd}(s)}{x_{even}(s)}$ is treated as an open-loop transfer function, and it is assumed that a unity-gain feedback loop is closed around this function, the resulting closed-loop poles will be exactly the factors of $x(s)$. To see this, write

$$\frac{\frac{x_{odd}(s)}{x_{even}(s)}}{1+\frac{x_{odd}(s)}{x_{even}(s)}} = \frac{x_{odd}(s)}{x_{even}(s)+x_{odd}(s)} = \frac{x_{odd}(s)}{x(s)}$$

All of these same statements, or course, also apply to $b(s)$, or to any other polynomial whose factors we wish to lie in the left half of the s-plane. These ideas are presently being investigated, and it is expected that the fully developed techniques will be presented in a later paper.

Another possibility which arises is that of changing the configuration shown in Figure 1. For instance, the compensation may be placed in the feedback path instead of the forward path. When this is done, the resulting closed-loop poles of the system will be the same, but the closed-loop zeros will be different. With the compensation in the feedback path, the plant zeros will still be closed-loop zeros, but now the compensation zeros will not. Instead, the closed-loop zeros occur at the location of the compensation poles. By placing part of the compensation in the forward path and part of it in the feedback path, one may obtain a variety of closed-loop transfer functions, all having the same poles.

Still another possibility regarding modifying the configuration involves the placement of smaller feedback loops inside of the over-all feedback loop. In this case, the compensation poles of the over-all system are identified with the closed-loop poles of one of the inner loops. The

present synthesis procedure may now be used to determine the components required inside the inner loop so that the closed-loop poles of the inner loop occur at the locations required for compensation poles of the over-all system.

Thus, the configuration assumed in Figure 1 is not unduly restrictive after all, because almost any configuration may be obtained by successive applications of the present method.

CONCLUSIONS

It has been shown how the locations of the closed-loop poles of a feedback control system may be specified and the necessary compensation may be found. This compensation does not require the use of cancellation or nullification of open-loop poles and zeros. The method is exact and involves nothing more difficult than the solution of a set of simultaneous equations. The next problem to be faced has been made quite clear by the present discussion. This next problem is how to specify the locations of the closed-loop poles so that the system meets the requirements without over-constraining anything. A method of getting from general specifications to mathematical constraints is needed.

The results achieved so far represent a step forward in the development of a new general approach to feedback system synthesis. The philosophy is to write down everything one knows about the system and the requirements on the system in the form of mathematical equations of constraint. These equations must be carefully formulated in order to insure that they will require the system to do what you want it to do without placing any unnecessary or impossible constraints on the system. This procedure yields a set of simultaneous equations concerning the poles and zeros of $\frac{N(s)}{D(s)}$. Some of these poles and zeros may be known at the outset and some of them will not. Then variables are added to the system in the form of poles and zeros of the compensation network until the number of variables equals the number of equations. When the equations are solved simultaneously, a solution is obtained which provides the poles and zeros of the resulting closed-loop system and the poles and zeros of the compensation.

Let us see what might be done about translating general system requirements into mathematical simultaneous equations. At present, very often the job the system is to do is stated by specifying the transfer function of the system in the form of a Bode plot of the frequency response, or a pole-zero plot, or else the time response to some transient input. Such a specification is almost always far too restrictive. It forces the system to do more than you really want. However, as a point of departure, one might assume that $W(s)$ is the ideal desired transfer function of the closed-loop system from its input to its output. $\frac{N(s)}{D(s)}$ is the actual closed-loop transfer function of the system.

One way of stating the requirements on the time response would be to specify an envelope rather than a specific function of time. Then, as long as the inverse Laplace transform of $\frac{N(s)}{D(s)}$ is a function which falls inside the envelope, the requirements are met. Or, by Fourier transform methods, the envelope in the time domain could be converted into envelopes of magnitude and phase measured along the $j\omega$ axis. Then, so long as $\frac{N(j\omega)}{D(j\omega)}$ falls inside these envelopes, the requirements will be met. There are also various means of approximating a given function with a polynomial, or a ratio of two polynomials, or a ratio of two polynomials subject to certain constraints. The goodness of fit may be prescribed subject to various weighting functions, mean-square error criteria, etc.

Of course, this part of the procedure still needs further study and at present it is necessary to work from both ends and try to patch things up in the middle. However, so much can be said about the form of $\frac{N(s)}{D(s)}$ that this process is more systematic than the usual trial-and-error procedure. The point is that the exact expression for $\frac{N(s)}{D(s)}$ is not assumed at the outset. Some adjustability is left so that compatibility with the fixed components in the system can be assured. Even though this compatibility requirement may place strong constraints

upon $\frac{N(s)}{D(s)}$, there are approximation techniques available which allow one to make $\frac{N(s)}{D(s)}$ approach the ideal desired $W(s)$ quite closely. For example, even if $N(s)$ were completely specified by the fixed components of the system, one could still obtain an acceptable $D(s)$ by such techniques so that $W(s)$ is well approximated.

Various methods along these lines have been developed in connection with the synthesis of passive networks, which might be applicable here. Any approach in which the problem is defined by a set of simultaneous equations and, in which there is room for additional equations of external constraint, should be quite applicable to the present problem.

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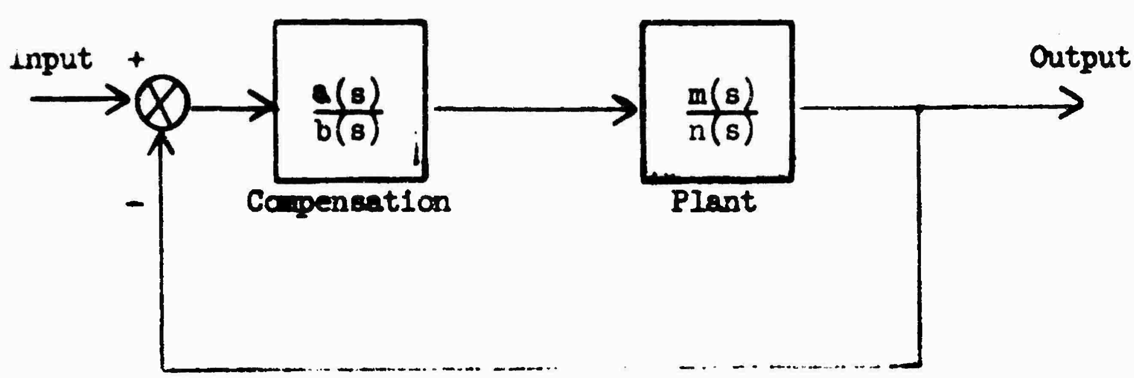
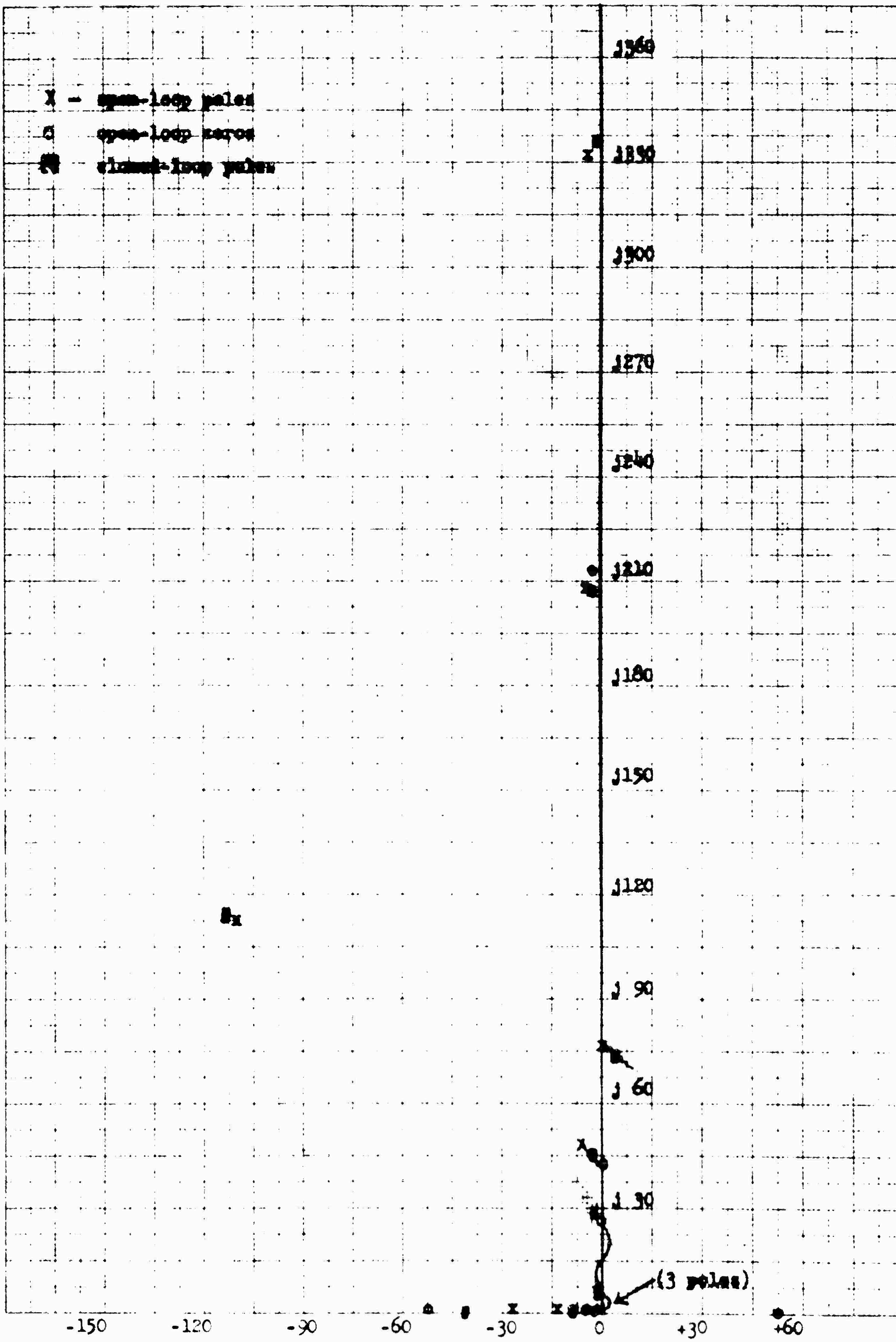


$$\frac{\text{Output}}{\text{Input}} = W(s)$$

FIGURE 1


X – open-loop poles
O open-loop zeros
closed-loop poles
j360
j330
j300
j270
j240
j210
j180
j150
j120
j 90
j 60
j 30
(3 poles)
-150
-120
-90
-60
-30
0
+30
+60

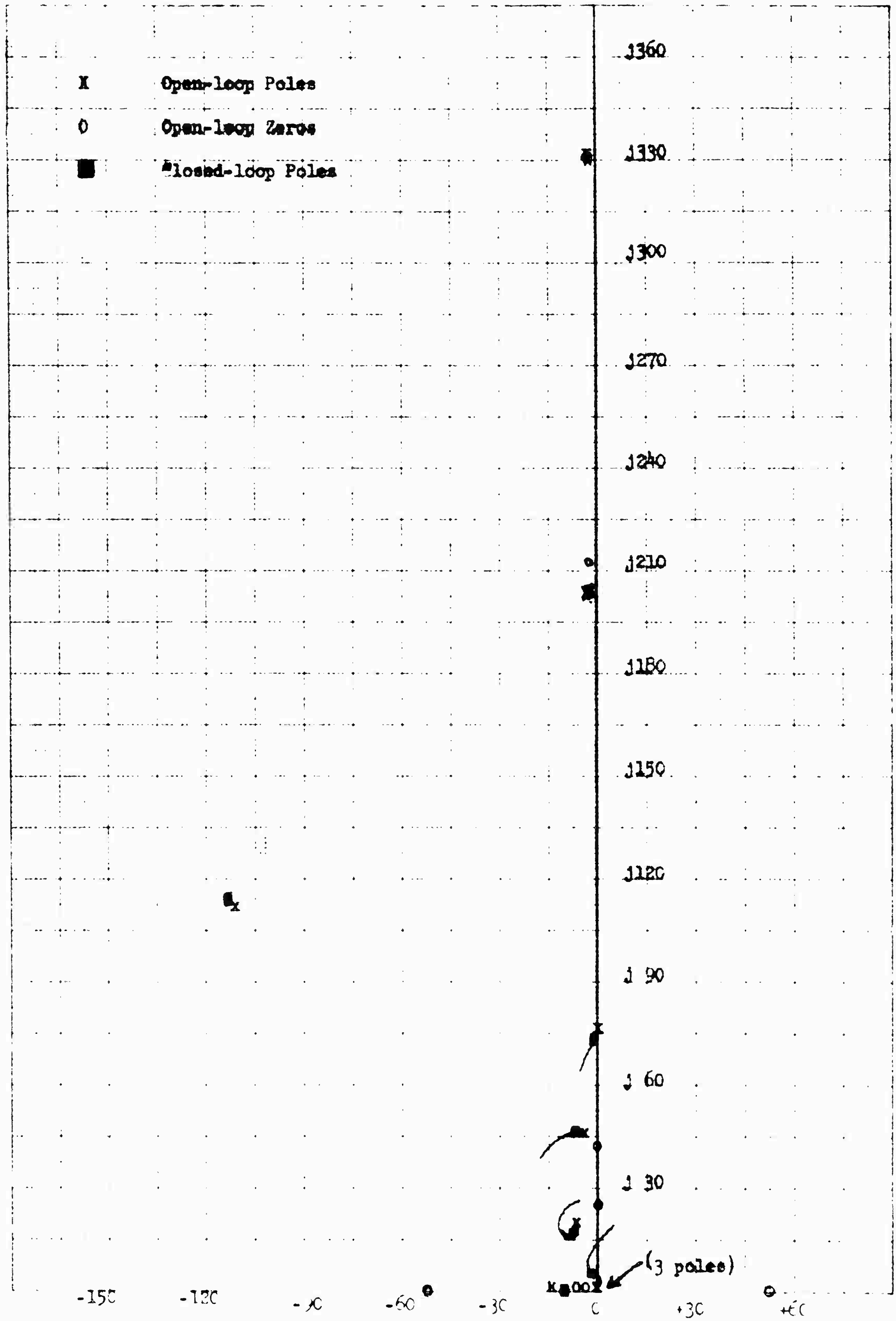

X Open-loop Poles
0 Open-loop Zeros
Closed-loop Poles
j360
j330
j300
j270
j240
j210
j180
j150
j120
j 90
j 60
j 30
(3 poles)
K=OO
-150
-120
-90
-60
-30
0
+30
+60

FIGURE 3

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